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What is a mean gravitational field?

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Abstract The equations of General Relativity are non-linear. This makes their averaging non-trivial. The notion of mean gravitational field is defined and it is proven that this field obeys the equations of General Relativity if the unaveraged field does. The workings of the averaging procedure on Maxwell's field and on perfect fluids in curved space-times are also discussed. It is found that Maxwell's equations are still verified by the averaged quantities but that the equation of state for other kinds of matter generally changes upon average. In particular, it is proven that the separation between matter and gravitational field is not scale-independent. The same result can be interpreted by introducing a stress-energy tensor for a mean-vacuum. Possible applications to cosmology are discussed. Finally, the work presented in this article also suggests that the signature of the metric might be scale-dependent too.

PACS. 04.20.Cv Fundamental problems and general formalism 04.40.Nr Einstein-Maxwell spacetimes, spacetimes with fluids, radiation or classical fields – 95.35.+d Dark matter (stellar, interstellar, galactic, and cosmological)

Notations

In this article, space-time indices running from 0 to 3 will be indicated by Greek letters. The metric signature will be $(+, -, -, -)$. I also have chosen, as a rule, *not* to use the so-called intrinsic notation in differential geometry and to stick to the notation standard in physics, which denotes every tensor by its components.

1 Introduction

Every observation of any system is necessarily finite i.e. the state of a given system is always represented by a finite number of variables, measured to a certain, finite precision. The nature of these variables, their number and the precision to which they are measured all vary with the retained experimental procedure. A given system is therefore generally susceptible of different, equally valid descriptions and building the bridges between those different descriptions is the task of statistical physics. Let me mention two historically famous examples.

A complete non-quantum theory of electromagnetic phenomena in continuous media can in principle be obtained by viewing the continuous media as made up of a great number of point-like charges; these define a charge and a charge current density which generate, via Maxwell equations, the electromagnetic field through which the particles interact. The real electromagnetic field varies therefore at both macroscopic and microscopic scales. To obtain from this description a reasonable model of what happens at macroscopic scales only, one usually defines a suitable averaging procedure (which has been historically introduced as a cutoff in Fourier-space for large 'wave-vectors' i.e. small wave-lengths); one thus obtains an average electromagnetic field and its average sources, which vary on macroscopic scales only [19,24]. Because the original field and the corresponding sources are linked by linear relations (i.e. Maxwell equations), their averages will be linked by the same relations. Maxwell equations are therefore valid both microscopically and macroscopically [24].

The other example is classical fluid dynamics, which is susceptible of at least three substantially different descriptions. Fundamentally, a non-quantum Galilean fluid can be viewed as a collection of point-like particles in interaction, the evolution of which obeys the laws of classical mechanics. On microscopic scales of the order of the mean free-path and the mean collision time between the particles, the state of a sufficiently diluted fluid can be described by a distribution function in one-particle phase-space and this distribution obeys a transport equation, typically the Boltzmann equation [17]. It turns out that the macroscopic behaviour of slowly varying solutions to the Boltzmann equation can be efficiently modelled by the Navier-Stokes system, the state of the fluid being now represented by a density field, a velocity field and

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another, thermodynamical field like the temperature [17]. Of course, the equations of motion have to be complemented by an equation of state, which links the various thermodynamical quantities to the density of the fluid; the equation of state must also be obtained by statistical analysis.

Let us spend some more time commenting on the Navier-Stokes model. A solution to the Navier-Stokes system corresponding to a turbulent flow will have a broad spectrum in Fourier space and so will probably a typical solution, varying notably on more than just one macroscopic scale [11,25]. Let us suppose for a moment, as an approximation, that a particular solution features only two variation scales, say L_1 and L_2, $L_1 \ll L_2$. Both of them are macroscopic but suppose that, for some reason, one is interested in finding an approximate description of the flow at scale L_2 only. One can introduce a suitable averaging procedure which furnishes, from the 'real' density, velocity field and temperature field of the fluid, new fields which vary only at scale L_2. The important point is that, generally speaking, these fields *do not* verify a system similar to the original Navier-Stokes system. In particular, the Navier-Stokes equation is in general *not* verified by the averaged fields [11]. This is due to the non-linearity of the original Navier-Stokes system. This kind of problem is not encountered when building a theory of electromagnetic phenomena in continuous media because Maxwell's equations are linear. The changes which appear in the Navier-Stokes hydrodynamical model are naturally linked to the centuries-old concept of eddy viscosity [11] and to the more modern one of renormalisation group [6]. Some interesting examples of actual analytical results and experimental confirmations can be found in [11].

In discussing both examples, I have tacitly assumed that the space-time was flat (or even non relativistic). Physicists now know how to lift this constraint and to implement the principles of statistical physics in a curved space-time background [18]. But a really general statistical physics in curved space-time should consider all degrees of freedom as susceptible of a statistical description. In other words, the gravitational field itself should be treated as a statistical quantity and one should be able, when needed, to average over various space-time curvatures as well as over different states of the 'matter'[1] present in space-time. The main difficulty comes from the non-linear character of the fundamental equations of General Relativity. This question is clearly of great theoretical or conceptual importance but its practical significance in astrophysical and cosmological applications cannot be overstressed either. For example, observational data seem to indicate that, on large scales, the universe is homogeneous and isotropic and, indeed, practically every calculation made in cosmology since the advent of General Relativity has assumed that it was possible to model the large-scale evolution of the universe by a spatially homogeneous isotropic solution to the equations of General Relativity [21,28]. Such a solution is meant to represent the mean gravitational field in the universe. But the whole procedure presupposes the possibility of defining a mean gravitational field which obeys the same evolution equations as the 'real' field, namely the fundamental equations of General Relativity. Since these equations are non-linear, it is not self-evident that this can be done. Even if one admits this task can nevertheless be accomplished by a suitable averaging procedure, taking averages a priori changes the equation of state of the matter present in the universe, e.g. its energy-momentum tensor, and to know how is also a clear priority of cosmology.

Various possible ways of averaging the geometry of space-time have already been proposed [4,5,12–14,20,31], but none of them seems fully satisfactory[2]. The aim of this article is to prove that there is a natural, physically meaningful way to average the Einstein gravitational field and that the procedure automatically ensures that the mean field also obeys the equations of General Relativity. As mentioned earlier, the equation of state for the matter present in space-time is changed by the averaging and this problem is also investigated. The material is organized as follows. Section 2 fixes the frame for the rest of the article by discussing the various notions of averaging that are commonly used in physics; of all the possibilities, the most general and practical one is certainly statistical averaging and the statistical approach is therefore retained for the remainder of this work. The statistical ensembles which are to be used in the general relativistic case are ensembles of space-times (or histories) and this is also discussed in Section 2. Section 3 proposes, as a case study, a brief review of how the selected averaging procedure works on Maxwell's equations in flat space-time. Section 4 presents the fundamental equations to be averaged. These are Einstein's equation, the equation expressing the compatibility of the metric and the connection and, finally, the equations of motion for the matter present in space-time. To keep the discussion focused, only two forms of matter are investigated, a (possibly charged) perfect fluid and Maxwell's electromagnetic field; the extension of the present work to other forms of matter is rapidly discussed in the conclusion. In Sections 5 and 6, the mean fields are defined as averages of various quantities and the evolution equations for these mean fields are also deduced. It is shown that defining the mean metric to be the average of the original metric and the mean connection to be the connection compatible with the mean metric leads to a mean gravitational field which obeys the equations of General Relativity.

Corresponding changes in the equations of motion and the equation of state of the matter present in space-time are also investigated in Section 6; in particular, the averaging of Maxwell's equations is not a trivial problem when the metric is itself a statistical variable. I propose a definition for the mean electromagnetic field that ensures that this field verifies Maxwell's equations. The averaging of the total stress-energy tensor and of the total charge 4-current are also discussed. Various notable conclusions

[1] This term is used here in its standard relativistic meaning; 'matter' thus represents all degrees of freedom but those of the gravitational field.

[2] See Section 7 for a full discussion of these references.

are reached; the first one is that the notion of perfect fluid is not scale-invariant. More generally, I also prove that the separation between gravitational field and matter is itself not scale-invariant. The same goes for the distinction between electric charge and electromagnetic field. These new effects are due to the non-linearity of the various equations of motion and appear only if the metric of space-time is itself a non-constant statistical variable. They disappear if the statistics are made in space-times corresponding to one and the same metric. These effects can also be interpreted by introducing a scale-dependent stress-energy and charge 4-current for the 'vacuum'; this point of view is less general than the preceding one but seems to be more in tune with the usual language of modern cosmology. Both interpretations are discussed in Section 6. As a conclusion, Section 7 reviews the work presented in this article and lists some possible extensions and applications. The differences between the approach presented in this article and those introduced in earlier references are also systematically discussed. It is shown that the previous averaging methods are either less general or not as physically and geometrically natural and meaningful as the framework developed in the present article.

2 General framework for the averaging procedure

There are essentially three different notions of averaging (or coarse graining) which are currently used in non-quantum physics. The first one consists in averaging a field by convoluting it with a specified 'window function' [19]. The second technique consists in adding variables to those of the original field, distinguishing then between the so-called slow and rapid variables; the average is then defined as an average over the rapid variables. This technique is most commonly used in dynamical systems theory, for example in studying turbulence [11]. Finally, one can also introduce Gibbs statistical ensembles and define the average as an average over the various members of these ensembles [15]. It is well known that, at least in Galilean physics, this third notion of average is more computationally convenient and more general than the other two, which it can accommodate as special cases. Thus, I have decided to work only with statistical averages for the rest of this article. Let us therefore spend some more time on defining them properly in a general relativistic context.

I would like to start this discussion with a brief review of the Galilean case. For any given scale conveniently represented by the symbol a, the evolution of a given system at scales larger than or comparable to a is defined by a set of differential equations which are verified by various variables, the nature and number of which depend on the scale a under consideration; in particular, the retained variables and their number may or may not be the same for various scales a.

Suppose now we have at our disposal a description of the system under consideration valid at scales larger than a given scale a, considered as 'microscopic' (but which may be in fact macroscopic with respect to some other scale) and that we now wish to obtain a correct, albeit less complete, description of the same system valid at scales larger than b, where b is much larger than a and defines the 'macroscopic' scale. In Galilean statistical physics, the natural notions to be used in this context are those of macroscopic and microscopic states, which have nearly obvious definitions. One can then define the mean value taken by an arbitrary microscopic quantity in a given macroscopic state of the system as the statistical average of this quantity over an ensemble of microscopic states which all correspond to this macroscopic state. This mean value is then identified to a macroscopically measurable quantity. In the Galilean case, Gibbs statistical ensembles are thus ensembles of micro-states corresponding to one and the same macro-state.

We now want to define a suitable relativistic generalization of these usual Galilean Gibbs ensembles. This is not a trivial problem. Indeed, it has already been noted that the concept of state itself is not a relativistic invariant notion [8]; statistical ensembles of states are therefore not relativistic invariants either and cannot be used in a consistent way to do statistical physics in Special or General Relativity. The remedy to this situation is to replace the concept of state by the notion of history and the ensembles of states by ensembles of histories, which specify the local state of the system at every point in space-time [8]. In [8], the notion of history has been used in the special relativistic context only. Let me therefore introduce it now in the general relativistic context.

In both Special and General Relativity, a macro-history is defined by the values taken by all macroscopic variables ('geometric' and 'non-geometric' ones) at every point of the space-time manifold. Similarly, a micro-history is defined by the values taken by all microscopic ('geometric' and 'non-geometric') variables at every point of space-time. A given macro-history is put into correspondence with an ensemble of micro-histories, over which any desired quantity can be averaged in a completely covariant manner. To put it slightly differently, each member of the statistical ensemble under consideration (i.e. each micro-history) is a space-time manifold; this manifold is endowed with a metric and the connection compatible with it (see below, Sect. 4.1.1); the metric and the connection of each member of the ensemble satisfy Einstein's equation, with a stress-energy tensor corresponding to a microscopic description of the matter in the space-time. The macro-history corresponds to a mean space-time. The central aim of this article is to explain precisely how the 'macroscopic' properties of this mean space-time emerge by averaging over the various members of the ensemble. More precisely, it will be shown that it makes physical and mathematical sense to define the mean metric over the mean space-time as the average of the metric over all members (space-times) in the ensemble. The mean metric is then compatible with a mean connection, which is also defined as the average of a certain 'microscopic' field. This field does not coincide with the 'microscopic' connection. Finally, it will be shown that the mean metric and connection satisfy Einstein's equations with a mean stress-energy tensor which also appears as a statistical average over the ensemble of space-times.

A final word about the notation: it will be convenient to introduce a parameter $\omega \in \Omega$ which labels the various members of a given statistical ensemble; averages over these ensembles will be denoted by using angular brackets.

3 A case study: Maxwell's equation in flat space-time

3.1 Basic equations

Maxwell's electromagnetism is a Yang-Mills gauge theory. The standard choice for the gauge field is the 4-potential A_μ, which appears as an element of the bundle cotangent to space-time [9]. In other words, the 4-potential is naturally a covariant 4-vector field. One can introduce the so-called 'curvature' form of A, characterized by the tensor F:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1}$$

F is naturally a twice covariant tensor field on space-time. A direct consequence of (1) is:

$$\partial_{[\alpha} F_{\mu\nu]} = 0, \tag{2}$$

where the brackets denote total antisymmetrization. This is the first pair of Maxwell's equations. The second pair links the tensor F to its source, which is characterized by a tangent vector field j. One has:

$$\eta^{\nu\alpha} \partial_\alpha F_{\mu\nu} = -4\pi \eta_{\mu\nu} j^\nu, \tag{3}$$

where η denotes the standard Minkowski metric. As noted by Maxwell himself, equation (3) implies charge conservation:

$$\partial_\mu j^\mu = 0. \tag{4}$$

An electromagnetic field acts on point charges and, more generally, on any current distribution. The motion of a point-like particle can be characterized by a trajectory, $X^\mu(s)$, parametrized by its proper-time s. The 4-velocity U of the particle is defined to be $U^\mu = dX^\mu/ds$ and is therefore a vector tangent to space-time, carrying naturally one contravariant index. On the contrary, what can be called the physical momentum of the particle is naturally an element of the cotangent space [18] and its components are linked to those of U by $P_\mu = m\eta_{\mu\nu}U^\nu$ (P_μ thus defined is *not* the canonical momentum of a point particle moving in a given electromagnetic field [2]). The equation of motion of a point charge q of mass m under the action of an electromagnetic field represented by F reads:

$$\frac{dP_\mu}{ds} = qF_{\mu\nu}U^\nu. \tag{5}$$

This equation can be cast into a somewhat different, more general form, by introducing the 4-current density j_P and the stress-energy tensor T_P associated to the point particle. In any Lorenz frame, one can write:

$$j^\mu_P(x) = \frac{q}{U^0(\sigma(x))}\, \delta^{(3)}(\mathbf{x} - \mathbf{X}(\sigma(x)))\, U^\mu(\sigma(x)), \tag{6}$$

where $\sigma(x)$ is the proper-time defined by $X^0(\sigma(x)) = x^0$. In the same way:

$${T_P}_\mu{}^\nu = \frac{1}{U^0(\sigma(x))}\, \delta^{(3)}(\mathbf{x} - \mathbf{X}(\sigma(x)))\, P_\mu(\sigma(x))\, U^\nu(\sigma(x)) \tag{7}$$

and (5) can be rewritten as [23]:

$$\partial_\nu {T_P}_\mu{}^\nu = F_{\mu\nu} j^\nu_P. \tag{8}$$

This equation is actually valid for an arbitrary charge distribution j associated to a stress-energy tensor[3] T; one thus has, quite generally:

$$\partial_\nu T_\mu{}^\nu = F_{\mu\nu} j^\nu. \tag{9}$$

Note that the choice of the mixed components of T is also quite natural in the present context. Indeed, one can define, for an arbitrary submanifold of dimension 3 in space-time, the following surface element [9]:

$$dS_\mu = \frac{1}{3!}\, \epsilon_{\mu\nu\alpha\beta}\, dx^\nu \wedge dx^\alpha \wedge dx^\beta, \tag{10}$$

which appears naturally as a covariant vector valued one-form; the quantities $T_\mu{}^\nu dS_\nu$ then represent the 4-momentum present in the 'surface' element dS; these constitute a covariant vector, like any naturally defined momentum.

3.2 The averaging

Let now $A(x, \omega)$, $F(x, \omega)$, $j(x, \omega)$ correspond to a certain ensemble of micro-histories, ω serving as a label for the different micro-histories in the ensemble. We want to define a mean 4-potential $\bar{A}(x)$, a mean tensor $\bar{F}(x)$, and a mean current density $\bar{j}(x)$ as averages over the considered ensemble.

Historically, the mean electromagnetic field has always been defined by averaging the effect of the real field on test particles. In more general terms, we will define the mean tensor $\bar{F}$ by averaging the effect of the real tensor $F(\omega)$ on test current distributions; by equation (9), this effect takes the form of a divergence of the stress-energy momentum tensor. Let us examine this point more closely, keeping in mind that the discussion will have to be extended to curved space-times (see Sect. 5.1).

Let $\mathcal{F}$ be an arbitrary foliation of space-time i.e. an arbitrary family of 3-dimensional submanifolds Σ's of space-time, independent of ω, such that every point x in space-time belongs to one and only one submanifold in the family; let $\Sigma(x)$ denote the submanifold containing

[3] For example, j and T can be the current and stress-energy tensor associated to a classical Klein-Gordon or Dirac field.

point x. Such a foliation $\mathcal{F}$ can be defined in both flat and curved space-times. In curved space-times, it can be used as a generalization of the family of constant-time hypersurfaces in Minskovski (flat) space-time. Note that the constant-time hypersurfaces in Minkovski space-time are all space-like but that nothing of the sort has been specified for the Σ's. This is because the space-like nature of a submanifold depends on the metric; imposing the Σ's to be necessarily space-like for all metrics in the statistical ensembles we will be considering in the next sections would thus be very restrictive. It turns out that this restriction is actually unnecessary and this is why it has not been retained in the general definition of $\mathcal{F}$.

Let now x be an arbitrary point in space-time. We want to fix to a quantity independent of ω the value taken by the current-distribution j_{test} of the test matter at points y on $\Sigma(x)$ around x. Let therefore ι be a field of vectors tangent to space-time, independent of ω, defined over some neighborhood D of x, $D \subset \Sigma(x)$. We will fix the test current-distribution on $\Sigma(x)$ around x by imposing:

$$j^{\mu}_{test}(y,\omega) = \iota^{\mu}(y) \tag{11}$$

for all ω and all $y \in D$. We also want to specify, in a similar way, the value taken by the stress-tensor of the test matter at all points in D. We therefore introduce a field $\theta^{\mu\nu}$, defined only on D and independent of ω; and we impose[4]:

$$T_{test\mu}{}^{\nu}(y,\omega) = g_{\mu\alpha}(y,\omega)\ \theta^{\alpha\nu}(y) \tag{12}$$

for all ω and all $y \in D$. This equation will not be used in the present Section but is also necessary to address the averaging problem in curved space-time. This is why the general form $g(y,\omega)$ has been retained in (12) instead of the simpler flat space-time Minkowski metric η.

Having fixed the value of the test current-distribution on D, one can then write (see (9)), for all ω and all $y \in D$:

$$\partial_\nu T_\mu{}^{\nu}(\omega)_{|y} = F_{\mu\nu}(y,\omega)\ \iota^{\nu}(y), \tag{13}$$

Equation (13), combined with the 'initial condition' (12), entirely determines the rate of change[5] of the stress-tensor of the test current distribution in all directions and, in particular, in the direction normal to $\Sigma(x)$ at y.

In a less general and more physically oriented language: if $\mathcal{F}$ is the family of constant-time hypersurfaces in Minkowski space-time and $x = (ct, \mathbf{x})$, the field ι represents the value of the test current distribution j_{test} at time t. The value of the stress-tensor T_{test} at time t is θ. Equation (13) determines the rate of change at time t of the stress-tensor T of the test current-distribution i.e. $\partial T/\partial t$.

[4] The reason for using these components and not other components of θ will become apparent in Section 5.1.

[5] Mathematically speaking, equation (13) determines a jet of tensor-fields at y.

Equation (13) can be written at point x and averaged over ω to give:

$$\langle \partial_\nu T_\mu{}^{\nu}(\omega)_{|x} \rangle = \langle F_{\mu\nu}(x,\omega) \rangle\ \iota^{\nu}(x), \tag{14}$$

which prompts the following definition for $\bar{F}$:

$$\bar{F}_{\mu\nu}(x) = \langle F_{\mu\nu}(x,\omega) \rangle. \tag{15}$$

Combining (15) and (1), one can write:

$$\bar{F}_{\mu\nu} = \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu, \tag{16}$$

where the mean 4-potential is simply defined as the average of the original A:

$$\bar{A}_\mu(x) = \langle A_\mu(x,\omega) \rangle. \tag{17}$$

The first pair of Maxwell's equations (2) is then automatically verified by $\bar{F}$ and the second pair (3) is also verified by the couple $(\bar{F}, \bar{j})$ where $\bar{j}$ is defined as the average of the original 4-current:

$$\bar{j}^{\mu}(x) = \langle j^{\mu}(x,\omega) \rangle. \tag{18}$$

This completes the averaging procedure of Maxwell's equations in flat space-time. As can be seen from the preceding discussion, these equations are clearly invariant under statistical average.

4 The equations to be averaged in curved space-time

4.1 A formalism for General Relativity

4.1.1 The gravitational field

General Relativity [9,30] views space-time as a 4-dimensional (real) manifold with coordinates (x^{μ}), $\mu = 0, 1, 2, 3$. This manifold is supposed to be endowed with two structures. The first one is a metric g, which is a second-order invertible tensor field. The metric defines the so-called line-element:

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}. \tag{19}$$

It can also be used to transform contravariant (tangent) vectors into covariant (cotangent) ones and vice-versa. More generally, it can be used to 'lower' indices or 'raise' them. In particular, if we denote the inverse to $g_{\mu\nu}$ by $g^{\mu\nu}$, one has:

$$g_{\mu\alpha} g^{\alpha\nu} = g^{\nu}_{\mu} = \delta^{\nu}_{\mu}, \tag{20}$$

where δ is the usual 4-dimensional Kronecker symbol. The metric of General Relativity is supposed to be Lorentzian, of signature $(+,-,-,-)$.

The second structure defined on space-time is an affine connection, which itself defines a so-called covariant derivative operator. The connection can be represented by a set of real-valued functions which will be noted $\Gamma^{\alpha}_{\mu\nu}$. This set of functions does *not* constitute a tensor field. The transformation law of these functions under a change of coordinate system is however perfectly known and reads, with obvious notations:

$$\Gamma^{\alpha'}_{\mu'\nu'} = \Gamma^{\alpha}_{\mu\nu} \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} + \frac{\partial^2 x^{\beta}}{\partial x^{\mu'} \partial x^{\nu'}} \frac{\partial x^{\alpha'}}{\partial x^{\beta}}. \quad (21)$$

We will denote by $\nabla(\Gamma)$ the covariant derivative (operator) associated to the functions $\Gamma^{\alpha}_{\mu\nu}$. One thus has, for example:

$$\nabla_\mu(\Gamma) V^\nu = \frac{\partial V^\nu}{\partial x^\mu} + \Gamma^{\nu}_{\mu\alpha} V^\alpha, \quad (22)$$

where V is an arbitrary (contravariant) vector field on space-time. The covariant derivative of vectors cotangent to space-time is defined by:

$$\nabla_\mu(\Gamma) W_\nu = \frac{\partial W_\nu}{\partial x^\mu} - \Gamma^{\alpha}_{\mu\nu} W_\alpha. \quad (23)$$

The transformation law (21) ensures that $\nabla_\mu(\Gamma)V^\nu$ and $\nabla_\mu(\Gamma)W_\nu$ are second order tensor fields on space-time. The covariant derivatives of tensors or spinors of arbitrary ranks can be defined and calculated in a similar way [30].

It is common to restrict the discussion to connections for which the covariant derivations of scalar fields with respect to different coordinates commute:

$$\nabla_\mu(\Gamma)\nabla_\nu(\Gamma)\,(\Phi) = \nabla_\nu(\Gamma)\nabla_\mu(\Gamma)\,(\Phi) \quad (24)$$

for any $\mu, \nu = 0, 1, 2, 3$ and an arbitrary scalar field Φ. One says then that the connection is torsion-free and this imposes that, in any coordinate system, the $\Gamma's$ are symmetric in their lower indices:

$$\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu}. \quad (25)$$

The connection can also be represented, given a metric g, by another set of functions, defined by [23]:

$$\Gamma_{\mu,\alpha\beta} = g_{\mu\nu} \Gamma^{\nu}_{\alpha\beta}, \quad (26)$$

which will be used later on. Note that these functions depend on both the connection and the metric.

The affine connection can be used to define a tensor called the curvature tensor *of the connection* or the Riemann tensor. The components of this tensor are, in an arbitrary coordinate system:

$$R^{\alpha}{}_{\mu\beta\nu} = \frac{\partial \Gamma^{\alpha}_{\nu\mu}}{\partial x^\beta} - \frac{\partial \Gamma^{\alpha}_{\beta\mu}}{\partial x^\nu} + \Gamma^{\alpha}_{\beta\sigma}\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\sigma}_{\beta\mu}. \quad (27)$$

To be noted is that this tensor depends only on the functions $\Gamma^{\mu}_{\alpha\beta}$ and not explicitly on the metric g. Only if the connection and the metric are related does the Riemann tensor become a functional of the metric.

The Riemann tensor can be contracted to give the Ricci tensor, also called the curvature tensor *of space-time*, the components of which are:

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}. \quad (28)$$

The use of the same letter to denote the Riemann and the Ricci tensors is naturally misleading but has been conserved in accordance with what is now common usage in relativistic physics.

In this article, we will consider that both the metric tensor and the connection code for gravitational effects. The fundamental equations of General Relativity are thus the evolution equations for the metric and the connection functions.

4.1.2 The fundamental equations of General Relativity

The first of these equations expresses the fact that the metric g is covariantly constant:

$$\nabla_\mu(\Gamma)\; g_{\alpha\beta} = 0 \quad (29)$$

for any $\mu, \alpha, \beta = 0, 1, 2, 3$. This implies that the Γ's associated to the connection can be expressed in terms of the metric tensor and its partial derivatives [30]:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\, g^{\alpha\beta} \left(\frac{\partial g_{\beta\nu}}{\partial x^\mu} + \frac{\partial g_{\beta\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta} \right). \quad (30)$$

Such a connection is called a metric connection and the expressions on the right-hand side of the preceding equation are called the Christoffel symbols of the metric g. Equation (30) thus states that the Γ's associated to a connection compatible with a given metric[6] are identical to the Christoffel symbols of this metric.

The functions $\Gamma_{\mu,\alpha\beta}$ associated to the same connection are:

$$\Gamma_{\mu,\alpha\beta} = \frac{1}{2} \left(\frac{\partial g_{\mu\alpha}}{\partial x^\beta} + \frac{\partial g_{\mu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu} \right). \quad (31)$$

Equation (31) will be of crucial importance in the averaging procedure for Einstein's gravitational field. Let us note, *en passant*, that the Ricci tensor of a metric connection is symmetric in its two indices.

The second basic equation of General Relativity is called Einstein's equation. It is usually interpreted by stating that the total energy-momentum tensor T of the matter present in space-time is the 'source' of gravitational phenomena; even though the mental image generated by this statement is basically correct, the use of the term 'source' is however misleading; indeed, when one considers General Relativity as a gauge theory [9,10,29], the tensor T which appears in Einstein's equation plays structurally a very different role from the role played by the current j in Maxwell's electromagnetism. The stress-tensor which appears in Einstein's equation is best defined through a variational approach as the partial derivative of the matter action with respect to the metric tensor g [30]. Since g

[6] A connection will be called compatible with a given metric g if the scalar product $g_{\mu\nu}v^\mu w^\nu$ of two vectors v and w remains unchanged if one parallel-transports these vectors along any curve.

is naturally a purely covariant tensor, the stress tensor appears naturally in Einstein's equation through its contravariant components $T^{\mu\nu}$; Einstein's equation equals this tensor T to a combination $\mathcal{E}(\nabla(\Gamma), g)$ of the Ricci tensor and of the metric tensor called the Einstein tensor and thus takes the form:

$$\mathcal{E}_{\mu\nu}(\nabla(\Gamma), g) \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \chi g_{\mu\alpha} g_{\nu\beta} T^{\alpha\beta}. \quad (32)$$

In this equation, R stands for the trace of the Ricci tensor and χ is a fundamental constant of nature, the precise expression of which depends on the chosen system of units. An exact and direct consequence of Einstein's equation is:

$$\nabla_\nu T^{\mu\nu} = 0, \quad (33)$$

which expresses the so-called conservation of energy and momentum. Hereafter, the traditional notation G will also be used for the Einstein-tensor $\mathcal{E}(\nabla(\Gamma), g)$.

4.2 The matter

The tensor T itself is the sum of various contributions corresponding to the different types of matter present in space-time. Each of these contributions generally depends also on the metric g. To keep the discussion focused, two kinds of matter only will be considered in the main part of this article; possible extensions will be discussed in the conclusion.

The first kind of matter which will be envisaged is an electromagnetic field. As in flat space-time, an electromagnetic field is traditionally represented by a 4-potential A_μ [30]. The tensor F is now defined by:

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu; \quad (34)$$

however, because of (23) and (25), definition (34) comes down to (1). The first pair of Maxwell's equation is again a direct consequence of (34) and reads now:

$$\nabla_{[\alpha} F_{\mu\nu]} = 0. \quad (35)$$

The second pair relates F to the charge 4-current j and takes the form:

$$g^{\nu\alpha} \nabla_\alpha F_{\mu\nu} = -4\pi g_{\mu\nu} j^\nu. \quad (36)$$

As in flat space-time, the second pair of Maxwell's equations implies charge conservation:

$$\nabla_\mu j^\mu = \frac{1}{\sqrt{-det\ g}}\, \partial_\mu \left(\sqrt{-det\ g} j^\mu \right) = 0. \quad (37)$$

The usual expression for the stress-energy tensor of the electromagnetic field in curved space-time reads:

$$T(A, g)^{\mu\nu} = \frac{1}{4\pi} \left(\frac{1}{4}\, F_{\alpha\beta} F_{\gamma\delta} g^{\alpha\gamma} g^{\beta\delta} g^{\mu\nu} - F_{\beta\alpha} F_{\gamma\delta} g^{\beta\mu} g^{\alpha\delta} g^{\gamma\nu} \right), \quad (38)$$

where care has been taken to make explicit the dependence of $T(A, g)^{\mu\nu}$ on both F and g. The dynamics of A and the expression (38) can be derived directly from a standard variational approach, to be found e.g. in [23] and [30].

Another form of matter traditionally coupled to Einstein's gravitational field is the so-called perfect fluid, which is represented by an energy-momentum tensor $\mathcal{T}$ of the form:

$$\mathcal{T}^{\mu\nu} = w u^\mu u^\nu - p g^{\mu\nu}, \quad (39)$$

where u represents the local 4-velocity of the fluid, w stands for its enthalpy density and p is the pressure field. Such a perfect fluid is also endowed with a conserved current J representing matter conservation [18]:

$$J^\mu = n u^\mu, \quad (40)$$

where n stands for the scalar particle density.

A perfect fluid can also have charges and be coupled to other interactions than gravity; only electromagnetic interactions will be considered in this article. If a perfect fluid is charged, its charge 4-current j is proportional to J, $j^\mu = qJ^\mu$, the ratio q representing the charge of the particles which constitute the fluid.

To make this study more specific, consider the classical example [1] of a charged perfect fluid evolving under the action of its own gravitational and electromagnetic fields[7]. The equations of motion for the system are, first, Maxwell's equations (35) and (36), which are the equations of motion for the electromagnetic field; second, relation (30), which ensures that the connection is compatible with the metric, and also Einstein's equation (32), with $T^{\mu\nu} = T(A, g)^{\mu\nu} + \mathcal{T}^{\mu\nu}$:

$$G_{\mu\nu} = \mathcal{E}_{\mu\nu}(\nabla(\Gamma), g) = \chi g_{\mu\alpha} g_{\nu\beta} \left[T(A, g)^{\alpha\beta} + \mathcal{T}^{\alpha\beta} \right], \quad (41)$$

$T(A, g)$ and $\mathcal{T}$ being respectively given by (38) and (39). The final equations of motion are the equations of motion for the perfect fluid itself. The first of these expresses the conservation of the particle current J:

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{-det\ g}}\, \partial_\mu \left(\sqrt{-det\ g} J^\mu \right) = 0; \quad (42)$$

it implies automatically the conservation of the charge 4-current j. The second equation can be condensed into [30]:

$$\nabla_\nu \mathcal{T}^\nu_\mu = F_{\mu\nu} j^\nu, \quad (43)$$

which is actually a direct consequence of (41) and (36). It is also an obvious generalization of (9) to curved space-time. The dynamics of a perfect fluid and the expression (39) for its stress-tensor can be derived from various variational principles; this is discussed in [7].

To the equations of motion must be added a so-called equation of state, which links the enthalpy density w and

[7] The more general case of several charged perfect fluids interacting with self-consistent gravitational and electromagnetic fields can be treated as a straightforward extension of the situation considered here and is of constant use in (plasma) Astrophysics and Cosmology.

the pressure p to the density n. In this article, the expression "equation of state" is used to designate something more general than what is commonly called equation of state. For example, (39) is considered to be part of the equation of state of a perfect fluid. In practice, this slightly uncommon usage should not generate any misunderstanding.

The six equations of motion are all non-linear. Their averaging is therefore not trivial and is the central topic of this article, fully discussed in the next section.

5 Definition of the mean gravitational and electromagnetic fields in curved space-time

5.1 Definition of the mean metric, of the mean connection and of the mean 4-potential

Let us consider a statistical ensemble of metrics and connections defined on space-time. As mentioned in Section 2, these ensembles are ensembles of histories, and not ensembles of states. The members of the ensemble will be labelled by the symbol $\omega \in \Omega$; we do not specify Ω since this is not necessary for what follows. To each ω corresponds a certain metric tensor $g(\omega)$, the connection $\Gamma(\omega)$ compatible with this metric, a certain electromagnetic field $A(\omega)$ and a certain perfect fluid flow. For each ω, all these fields satisfy the equations presented in the preceding section. All members of the ensemble correspond to the same macroscopic history of the space-time manifold, in particular to a same given mean metric $\bar{g}$, mean connection $\bar{\Gamma}$ and mean electromagnetic field $\bar{A}$. The problem is to define the mean quantities as averages over the statistical ensemble and to deduce from the retained definitions the evolution equations for the mean fields. In what follows, we will suppose that all manifolds in the ensemble admit a common atlas i.e. we can find a set of coordinate systems common to all members of the ensemble.

Equation (30) is somewhat simpler than the others because it only involves the metric and the connection. We will start therefore by examining if it is possible to find a definition for the mean metric and the mean connection which ensures that these quantities are compatible.

Since equation (30) is non-linear, it is obvious that defining the mean metric to be the average of the metric and the mean connection to be the average of the connection does not solve the problem. In other words, the average of the metric and the average of the Christoffel symbols are not generally compatible. Worse than that, the average of the connection does indeed furnish a connection on space-time, but there is no reason why this average should be compatible with any metric. The solution to the problem is given by equation (31), which is actually a linearization of (30). Indeed, (26) is linear in both g and $\Gamma_{\mu,\alpha\beta}$. This suggests that the mean metric is the average of the metric:

$$\bar{g}_{\mu\nu}(x) = \langle g_{\mu\nu}(x,\omega)\rangle \tag{44}$$

and that:

$$\bar{\Gamma}_{\mu,\alpha\beta}(x) = \langle \Gamma_{\mu,\alpha\beta}(x,\omega)\rangle . \tag{45}$$

These definitions naturally ensure that the mean connection $\nabla(\bar{\Gamma})$ is compatible with the mean metric and, assuming the mean metric $\bar{g}$ to be invertible, the usual functions representing the mean connection are then given by (see (26)):

$$\bar{\Gamma}^{\mu}_{\alpha\beta}(x) = \bar{g}^{\mu\nu}(x)\bar{\Gamma}_{\nu,\alpha\beta}(x), \tag{46}$$

with:

$$\bar{g}_{\mu\rho}\bar{g}^{\rho\nu} = \delta^{\nu}_{\mu}. \tag{47}$$

Using (45), this can be rewritten as:

$$\bar{\Gamma}^{\mu}_{\alpha\beta}(x) = \bar{g}^{\mu\nu}(x)\langle \Gamma_{\nu,\alpha\beta}(x,\omega)\rangle \tag{48}$$

or:

$$\bar{\Gamma}^{\mu}_{\alpha\beta}(x) = \bar{g}^{\mu\nu}(x)\left\langle g_{\nu\rho}(x,\omega)\Gamma^{\rho}_{\alpha\beta}(x,\omega)\right\rangle . \tag{49}$$

Let me stress again that, the $\bar{\Gamma}^{\mu}_{\alpha\beta}$'s being *not* the averages of the original $\Gamma^{\mu}_{\alpha\beta}$'s, the mean connection cannot be considered as the average of the original connection.

The choice of definition (45) is supported by the investigation of the equation of motion for test matter present in space-time. This will also yield a natural definition for the mean electromagnetic 4-potential $\bar{A}$.

Equation (43) has been derived for a perfect fluid but it is actually valid for any kind of charged matter interacting solely with Einstein's gravitational field and the electromagnetic field F. This equation can be rewritten as [23]:

$$\frac{1}{\sqrt{-det\ g}}\,\partial_\nu\left(\sqrt{-det\ g}\ T_\mu{}^{\nu}(\omega)\right)_{|x} = \Gamma_{\beta,\alpha\mu}(x,\omega)T^{\alpha\beta}(x,\omega) + F_{\mu\nu}(x,\omega)j^{\nu}(x,\omega), \tag{50}$$

where T and j stand for an arbitrary stress-energy tensor and an arbitrary 4-current density. Equation (50) is the curved space-time generalization of (9). (50) expresses that the total 4-momentum of charged matter is not a conserved quantity and that it varies because the matter exchanges momentum with the gravitational field (first term on the right-hand side of (50)) and with the electromagnetic field (second term on the right-hand side of (50)).

Let us now consider the motion of test matter under both gravitational and electromagnetic fields. One can write, with the notations introduced previously in Section 3.2 (see (12) and (11)):

$$\frac{1}{\sqrt{-det\ g}}\,\partial_\nu\left(\sqrt{-det\ g}\ T_\mu{}^{\nu}(\omega)\right)_{|x} = \Gamma_{\beta,\alpha\mu}(x,\omega)\theta^{\alpha\beta}(x) + F_{\mu\nu}(x,\omega)\iota^{\nu}(x). \tag{51}$$

This supports definitions (44) and (45); it also suggests:

$$\bar{F}_{\mu\nu}(x) = \langle F_{\mu\nu}(x,\omega)\rangle; \tag{52}$$

because of equation (34), which actually reduces to (1), this leads to the definition:

$$\bar{A}_{\mu}(x) = \langle A_{\mu}(x,\omega)\rangle. \tag{53}$$

These definitions of the mean 4-potential and of the mean tensor $\bar{F}$ ensure that the first pair of Maxwell's equations, equation (35), is automatically verified; this is so because (35) actually reduces to (2).

One is also led naturally to definitions (44) and (45) for the mean metric and connections by investigating the propagation of a test electromagnetic field in curved space-time. Indeed, for a Maxwell field interacting only with Einstein's gravitational field, one can write [23], with obvious notations:

$$\frac{1}{\sqrt{-det\ g}}\ \partial_\nu \left(\sqrt{-det\ g}\ T_\mu{}^{\nu}(A)(\omega)\right)_{|x} = \Gamma_{\beta,\alpha\mu}(x,\omega)\theta^{\alpha\beta}(A)(x), \quad (54)$$

which leads to the same definition of the mean gravitational field.

5.2 Some general comments

Some very general comments and a partial summing up are in order before establishing the equations of motion verified by the mean fields.

Our definitions of the mean gravitational and electromagnetic fields can receive two, very different motivations.

The first one is rather formal. The retained definitions are indeed the only ones which ensure that the mean connection is compatible with a metric and that the mean tensor $\bar{F}$ does derive from a 4-potential $\bar{A}$.

The second motivation comes from the analysis of the average motion of test matter in the gravitational and electromagnetic fields. The 4-momentum of a point particle is naturally a cotangent vector [18]; the equation of motion of a point mass is thus best represented by a differential equation fixing the variation of the 4 covariant momentum components. The generalization to continuous media physics and to field theory of the 4 covariant momentum components of a point-like particle is the set of the stress-energy tensor's mixed components [23]; and one can read off the expression for the fields from the differential equation (50) expressing the non-conservation of T_μ^ν because of its interactions with these fields.

To obtain from such an analysis the same expression for the mean fields as the ones already obtained via the other route, it is necessary to represent test matter by a charge-current (contravariant) 4-vector j^μ and by the *contravariant* components $T^{\mu\nu}$ of T. Both choices are best understood from a variational point of view. The 4-current j is usually defined as the functional derivative of the action with respect to the 4-potential A [3]. The gauge structure [9] of Maxwell's theory dictates that A is naturally a field cotangent to space-time. The 4-current j therefore appears as a vector field tangent to space-time represented by its contravariant components j^μ. These represent the source of the electromagnetic field and are also used to probe the field; hence the last term in (51).

As mentioned above, the stress-tensor T does not have the same status as j i.e. it is not the derivative of the action with respect to the connection Γ. It is nevertheless at the origin of gravitational phenomena as described by General Relativity. As such it is best defined as the functional derivative of the action with respect to the metric tensor [30] and appears therefore naturally in Einstein's equation through its *contravariant* components $T^{\mu\nu}$. These components are thus the natural probes for the gravitational field; they are used as such in (51).

6 The equations of motion for the mean fields and their interpretations

6.1 The equations of motion

Two equations have already been determined. The first one expresses the compatibility of the mean connection with the mean metric:

$$\nabla_\alpha(\bar{\Gamma})\ \bar{g}_{\mu\nu} = 0 \quad (55)$$

for all α, μ and ν. The other one states that the mean tensor $\bar{F}$ derives from the mean 4-potential $\bar{A}$:

$$\bar{F}_{\mu\nu} = \nabla_\mu(\bar{\Gamma})\bar{A}_\nu - \nabla_\nu(\bar{\Gamma})\bar{A}_\mu \quad (56)$$

and one has therefore:

$$\nabla_{[\alpha}(\bar{\Gamma})\bar{F}_{\mu\nu]} = 0 \quad (57)$$

for all α, μ and ν. This comes from the extact cancellation of all non-linear terms in both (56) and (57).

Let us now derive Einstein's equation for the mean gravitational field. One can define:

$$\bar{G}_{\mu\nu} = \mathcal{E}_{\mu\nu}(\nabla(\bar{\Gamma}),\bar{g}) = \bar{R}_{\mu\nu} - \ \frac{1}{2}\ \bar{R}\ \bar{g}_{\mu\nu}, \quad (58)$$

the Einstein tensor associated to the mean metric and connection. This tensor is generally different from the average $\langle G(\omega)\rangle$. Introducing:

$$\Delta G_{\mu\nu}(x) = \langle G_{\mu\nu}(x,\omega)\rangle - \bar{G}_{\mu\nu}(x), \quad (59)$$

one can write:

$$\bar{G}_{\mu\nu} + \Delta G_{\mu\nu} = \chi\langle g_{\mu\alpha}(\omega)g_{\nu\beta}(\omega)T^{\alpha\beta}(\omega)\rangle. \quad (60)$$

Let us define the mean stress-tensor $\bar{T}$ by:

$$\bar{T}^{\alpha\beta} = \bar{g}^{\alpha\mu}\bar{g}^{\beta\nu}\left[\langle g_{\mu\rho}(\omega)g_{\nu\sigma}(\omega)T^{\rho\sigma}(\omega)\rangle - \ \frac{1}{\chi}\ \Delta G_{\mu\nu}\right]; \quad (61)$$

one can then write:

$$\bar{G}_{\mu\nu} = \mathcal{E}_{\mu\nu}(\nabla(\bar{\Gamma}),\bar{g}) = \bar{R}_{\mu\nu} - \ \frac{1}{2}\ \bar{R}\ \bar{g}_{\mu\nu} = \chi\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}\bar{T}^{\alpha\beta}, \quad (62)$$

which has exactly the same form as (32). As a consequence, the mean stress-tensor $\bar{T}$ is conserved:

$$\nabla_\mu(\bar{\Gamma})\bar{T}^{\mu\nu} = 0. \quad (63)$$

The second pair of Maxwell's equation for the mean electromagnetic field can be obtained in a similar manner. One can introduce:

$$\Delta j_\mu(x) = \langle g^{\nu\alpha}(x,\omega)(\nabla_\alpha(\Gamma(\omega))F_{\mu\nu}(\omega))_{|x}\rangle - \bar{g}^{\nu\alpha}(x)\big(\nabla_\alpha(\bar{\Gamma})\bar{F}_{\mu\nu}\big)_{|x} \quad (64)$$

and define the mean charge 4-current $\bar{j}$ by:

$$\bar{j}^\mu = \bar{g}^{\mu\nu}\langle g_{\nu\alpha}(\omega)j^\alpha(\omega)\rangle - \Delta j^\mu, \quad (65)$$

so that:

$$\bar{g}^{\nu\alpha}\nabla_\alpha(\bar{\Gamma})\bar{F}_{\mu\nu} = \bar{g}_{\mu\nu}\bar{j}^\nu. \quad (66)$$

This is naturally the second pair of Maxwell's equations; (66) implies that $\bar{j}$ is strictly conserved:

$$\nabla_\mu(\bar{\Gamma})\bar{j}^\mu = 0. \quad (67)$$

Let us note that Δj vanishes identically if $g(\omega)$ does not depend explicitly on ω i.e. if the statistical average is done on a fixed curved space-time background.

Let us now investigate in more details some consequences of the preceding equations if the matter present in space-time before averaging appears as a charged perfect fluid interacting with a self-consistent electromagnetic field.

Let us assume that $\bar{j}$ is time-like i.e. $\bar{g}_{\mu\nu}\bar{j}^\mu\bar{j}^\nu > 0$. One can then define a charge 4-velocity u^μ by:

$$u^\mu = \frac{\bar{j}^\mu}{(\bar{j}\cdot\bar{j})^{1/2}} \quad (68)$$

and a density n by:

$$n = \frac{1}{q}\,(\bar{j}\cdot\bar{j})^{1/2}, \quad (69)$$

where q is the charge of the point-particles constituting the fluid. But, for a perfect fluid, it is possible to define another mean-density and mean-velocity. These are obtained by suitably averaging the matter 4-current J. Indeed, because $J(\omega)$ is conserved for each ω, one can define:

$$\bar{J}^\mu = \frac{1}{\sqrt{-det\ \bar{g}}}\left\langle\sqrt{-det\ g(\omega)}J^\mu(\omega)\right\rangle, \quad (70)$$

where $det\ \bar{g} < 0$ has been assumed. $\bar{J}$ is automatically time-like; as $\bar{j}$, it is also conserved. Indeed, one can write:

$$\begin{aligned}\nabla_\mu(\bar{\Gamma})\bar{J}^\mu &= \frac{1}{\sqrt{-det\ \bar{g}}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-det\ \bar{g}}\bar{J}^\mu\right)\\ &= \frac{1}{\sqrt{-det\ \bar{g}}}\frac{\partial}{\partial x^\mu}\left\langle\sqrt{-det\ g(\omega)}J^\mu(\omega)\right\rangle\\ &= \frac{1}{\sqrt{-det\ \bar{g}}}\left\langle\frac{\partial}{\partial x^\mu}\sqrt{-det\ g(\omega)}J^\mu(\omega)\right\rangle\\ &= \frac{1}{\sqrt{-det\ \bar{g}}}\left\langle\sqrt{-det\ g(\omega)}\left(\nabla_\mu(\Gamma(\omega))J^\mu(\omega)\right)\right\rangle\\ &= 0, \end{aligned} \quad (71)$$

since $J(\omega)$ is conserved for each ω.

$\bar{J}$ can be associated to a 4-velocity U and a particle density N respectively defined by:

$$U^\mu = \frac{\bar{J}^\mu}{(\bar{J}\cdot\bar{J})^{1/2}} \quad (72)$$

and

$$N = (\bar{J}\cdot\bar{J})^{1/2}. \quad (73)$$

In general, $\bar{J}$ is not identical to $\bar{j}/q$ and, therefore, U is different from u and N is different from n. One can always define a mean stress-tensor $\bar{\mathcal{T}}$ by:

$$\bar{\mathcal{T}}^{\mu\nu} = \bar{T}^{\mu\nu} - T(\bar{A},\bar{g})^{\mu\nu} \quad (74)$$

and $\bar{\mathcal{T}}$ will automatically verify the equivalent of (43):

$$\nabla_\nu(\bar{\Gamma})\bar{\mathcal{T}}^\nu_\mu = \bar{F}_{\mu\nu}\bar{j}^\nu; \quad (75)$$

but $\bar{\mathcal{T}}$ will generally not have a simple expression in terms of u, n, U and N. In particular, $\bar{\mathcal{T}}$ will generally not take the simple form characteristic of a perfect fluid. The precise form of $\bar{\mathcal{T}}$ depends on the physics happening at the scales which have been averaged upon and, therefore, of the precise problem under consideration. No general conclusion can be made at this point.

6.2 Possible interpretations and some consequences of the results

Equations (61) and (65) show that the 'small' scales have a non-trivial effect on the large ones and modify in a possibly very complicated manner the equation of state of the matter present in space-time. In the presence of a gravitational field, the equation of state is thus generally scale-dependent. The situation is actually even more intricate. Indeed, the mean stress-tensor $\bar{\mathcal{T}}$ is made up of two apparently very different contributions. The first one is directly linked to a mean value involving the 'original' stress-tensor T, while the other one involves a seemingly purely geometrical term ΔG. This suggests that the separation between matter and gravitational field is itself scale-dependent, even at the level of non-quantum classical General Relativity. There is another interesting interpretation of equation (61). Suppose we have experimental or observational access to $\langle g_{\mu\rho}(\omega)g_{\nu\sigma}(\omega)T^{\rho\sigma}(\omega)\rangle$. Suppose we do not want to make the notion of matter scale-dependent but declare, for example, that points where $\langle g_{\mu\rho}(\omega)g_{\nu\sigma}(\omega)T^{\rho\sigma}(\omega)\rangle$ vanish represent the 'vacuum'. At such points, $\bar{\mathcal{T}}$ does not vanish and $-1/\chi\Delta G$ at these points might thus be considered the stress-energy of the vacuum. This point of view might appear more physical, especially in a cosmological context [22], where an estimate of $\langle g_{\mu\rho}(\omega)g_{\nu\sigma}(\omega)T^{\rho\sigma}(\omega)\rangle$ can probably be obtained observationally [21,28]. The other interpretation however seems both more general and natural, at least from a purely theoretical stance.

Similar comments can be made upon equation (65): the mean charge 4-current $\bar{j}$ is also made up of two contributions. The first one is an average of a quantity involving directly j; but the term Δj depends explicitly only on the gravitational and electromagnetic fields, and not the current-distribution. The separation between electromagnetic field and electric charge current is therefore scale-dependent too. One can also interpret the $-\Delta j$ contribution to $\bar{j}$ as a charge 4-current of the 'vacuum'. Again, this point of view seems less general than the preceding one.

Let us also note that both effects under discussion disappear if the statistical averaging is made over an ensemble where the metric is constant i.e. independent on ω.

A final important comment, concerning possible cosmological applications. Let us suppose that, in a given reference frame, what happens at the scales which are averaged upon is sufficiently 'random' to make the averaged physics spatially homogeneous and isotropic. This is of course exactly the situation encountered in modern cosmology [27,28]; the Universe is certainly not homogeneous nor isotropic on 'small' scales; but there apparently exists a reference-frame, the so-called cosmic rest-frame, in which the Universe is homogeneous and isotropic on scales comparable to the Hubble-length.

Let this particular reference frame be represented by a 4-velocity V^μ, normed to unity. Since the description at large scales is supposed to be homogeneous and isotropic, the mean stress-energy tensor necessarily takes the form [16]:

$$\bar{T}^{\mu\nu} = \bar{w} V^\mu V^\nu - \bar{p} \bar{g}^{\mu\nu}. \quad (76)$$

If $\langle g_{\mu\rho}(\omega) g_{\nu\sigma}(\omega) T^{\rho\sigma}(\omega) \rangle$ is one of the fields accessible to observation, it is part of what I called 'the averaged physics' and therefore, by hypothesis, takes a form similar to $\bar{T}$:

$$\langle g_{\mu\rho}(\omega) g_{\nu\sigma}(\omega) T^{\rho\sigma}(\omega) \rangle = W V^\mu V^\nu - P \bar{g}^{\mu\nu}. \quad (77)$$

A direct consequence of both preceding equations is that the tensor ΔG also takes the simple perfect-fluid form:

$$\Delta G^{\mu\nu} = w_{vac} V^\mu V^\nu - p_{vac} \bar{g}^{\mu\nu}, \quad (78)$$

where the notation reflects the fact that, as already mentioned, cosmology would interpret w_{vac} and p_{vac} as the enthalpy-density and pressure of the 'vacuum'. There does not seem however to be any constraints on the values and signs of these quantities; in other words, there does not seem to be any constraint on the equation of state of the 'vacuum' in this approach. The natural question raised by these comments concerns the cosmological constant i.e. can one have $\Delta G = \Lambda g$? Further work is obviously needed to answer this very interesting question.

7 Conclusion

7.1 Summary

The general relativistic equations fixing the evolution of the gravitational field and of the matter present in space-time are non-linear. Their averaging is therefore not trivial. In this article, I have proposed a general procedure which permits to construct mean gravitational and matter fields as averages over statistical ensembles of space-time histories. The procedure ensures that the mean gravitational and electromagnetic fields obey the equations of General Relativity and Maxwell's equations in curved space-time, exactly as the unaveraged quantities. The mean metric (in purely covariant form) turns out to be the average of the original metric and the mean connection is the connection compatible with the mean metric. These definitions for the mean gravitational field have been given an interpretation in terms of mean evolution experienced by test systems.

The mean electromagnetic field has also been defined. The mean (covariant) 4-potential is the average of the original (covariant) 4-potential and the mean electromagnetic tensor $\bar{F}$ (in purely covariant form) is then the average of the original electromagnetic tensor F. These definitions have also been given an interpretation in terms of mean motion experienced by test current distributions.

The mean gravitational and electromagnetic fields obey equations that are similar to those obeyed by the unaveraged field. The price to pay for this is a change in the equation of state for matter (Maxwell's field excepted). In particular, matter which can be modelled as a perfect fluid before the averaging procedure does not generally appear as a perfect fluid after the averaging procedure has been applied. Things are actually even more complex. The mean stress-energy tensor is made up of two contributions. One is the average of a quantity which depends on both the unaveraged metric and the unaveraged stress-energy tensor. The other contribution, however, appears to be a 'purely geometrical' quantity, involving explicitly only the gravitational field. This can be interpreted in two different manners. A first possibility is to consider that the separation between the gravitational field and the matter which curves space-time is scale-dependent. This seems to be the most general point of view. The other interpretation is only useful if one has experimental or observational access to the first contribution to the mean stress-tensor, i.e. the one which depends directly on the unaveraged stress-energy tensor. It then makes sense to interpret this term as the only one associated to 'real' matter and to interpret the other contribution as a vacuum stress-energy tensor. This kind of interpretation is perhaps most useful in cosmology [21,28].

Similar remarks apply to the electric charge 4-current. The mean 4-current has two contributions. One is the average of a quantity which depends directly on the unaveraged 4-current. The other term however depends explicitly only on the gravitational and electromagnetic fields. Here again, one can say that the separation between electromagnetic field and charge current is scale-dependent or introduce the concept of a charge 4-current associated to the vacuum.

Both these new effects are due to the non-linearity of the evolution equations. They disappear if the metric of space-time is the same for all histories in the statistical ensemble.

Before discussing some directions in which this work can be developed, let me point out one of its hitherto unmentioned theoretical assets. The whole averaging procedure presented in this article can be viewed as a machine to produce exact unknown solutions to the

Einstein-Maxwell system of equations from any given known solution; these new solutions describe in a coarse-grained manner the same 'physics' as the known ones, but they are nevertheless new. The study of some of these new solutions (for example, coarse-grained Reisner-Nordström black-holes) should prove physically interesting and may also contribute to a better understanding of the averaging procedure itself.

As a conclusion to this summary, it is interesting to mention that the philosophy underlying the definition of the mean stress-energy tensor $\bar{T}$ presents some common points with the ideas governing the construction of an effective stress-energy pseudo-tensor for the gravitational field itself. Let me be more precise about this.

It has been argued in Section 5 that the mean gravitational field is represented by the metric $\bar{g}$ which is the mean value of the original metric $g(\omega)$. One is therefore interested in the Einstein tensor $\bar{G}$ associated to this metric and to the connection $\nabla(\bar{\Gamma})$ compatible with it. This tensor naturally does not coincide with the average of the original Einstein tensor $G(\omega)$ which appears naturally upon averaging the original Einstein's equation. One therefore singles out the contribution of $\bar{G}$ to the averaged Einstein's equation and all the other contributions are transferred to the right-hand side of the equation and define together the mean (or effective) stress-energy tensor $\bar{T}$.

In a similar spirit, one can single out the contribution to Einstein's equation linear in the metric and all other terms grouped together then define an effective stress-energy pseudo-tensor for the gravitational field [23,26]. All this being said, there are naturally important differences between both procedures. Perhaps the most obvious one is that the statistical averaging defined in this article is a perfectly covariant procedure, as any statistical averaging should be, whereas the definition of a stress-energy pseudo-tensor for the gravitational field is not (and cannot be) a covariant operation.

7.2 Comparison with previous work

As mentioned in the Introduction, several other ways of averaging the geometry of space-time have already been proposed in the literature; let me now discuss at some length at least the most recent and/or general of these other approaches and compare them with the present work.

References [4,5,12–14,20] all belong to the same family. Strictly speaking, references [4,5] do not belong to the same group as [12–14,20], but both groups share sufficient basic characteristics to make a common discussion mandatory. First of all, the average procedure retained by [4,5,12–14,20] is not a statistical averaging but a three-dimensional spatial averaging. More importantly, the very conception of the averaging procedure is tailor-made for the Friedmann-Robertson-Walker (FRW) cosmologies and an extension to other space-times seems out of the question. In particular, none of these references proposes a general definition of the mean metric or of the mean connection, which can then be applied to particular space-times. Indeed, only the mean scale-factor of a FRW universe is introduced, and it is defined only in terms of the mean energy density of the universe, as measured on three-dimensional submanifolds of space-time where 'space' appears to be homogeneous and isotropic. All this is made with the aid of the so-called $3+1$ formalism, and no (foliation independent) manifestly covariant treatment is proposed.

Let me characterize further some individual references in both families. Of the three works by Futamase, [14] is clearly the most developed one. Unfortunately, it only deals with vanishingly small perturbations to the FRW universe. The work of Kasai [20] substantially lifts this constraint, but it is still only applicable to FRW cosmologies. Moreover, the only matter envisaged in [20] is dust.

Reference [4] again deals only with dust. The formalism is extended to perfect fluid cosmologies in [5]; the case where the perfect fluid is made up of radiation is discussed in Section 4.3 of [5], but an average electromagnetic 4-potential is unfortunately not defined, and average Maxwell equations are consequently not derived. Moreover, [5] suffers from all of the drawbacks mentioned earlier. In particular, reference [5] only defines the average of scalar fields and does not shed any light on how to define the average of general tensor fields (see Sect. 3.1 of [5]).

The framework introduced in the present work does not suffer from the same limitations. Indeed, the defined averaging procedure is based on the very general notion of statistical ensembles, properly extended to take into account the gravitational degrees of freedom. The statistical ensembles, being ensembles of space-times, are scalar and the whole procedure is not only covariant, but also manifestly covariant. The average metric, connection, curvature tensor, stress-energy tensor and electromagnetic 4-potential have all been defined in a very general context. The average equations of motion for matter have been investigated and, in particular, Maxwell equations for the mean electromagnetic field have been derived. Moreover, the whole framework is applicable to arbitrary physical situations, and not only to cases where the average space-time is a FRW universe.

Let us now discuss reference [31]. This reference stands on its own and is in many ways closer to the present work than all the other references I have just discussed. The introduced average procedure is admittedly not of a real statistical nature, but it is a manifestly covariant averaging over space-time domains of a given four-dimensional (scalar) volume. This *de facto* makes the whole framework applicable to arbitrary space-times, and not to FRW cosmologies only. There is however a crucial difference between [31] and the present work. At the beginning of Section 4, reference [31] states: "The average... of the microscopic Levi-Civita connection $\gamma^{\alpha}_{\beta\gamma}$ is supposed to be Levi-Civita's connection of the averaged space-time[8]." But the author does not justify this supposition, never showing that it is reasonable and makes

[8] Reference [31] uses the notation γ for the Christoffel symbols, and not Γ.

geometrical and/or physical sense. It is simply a supposition underlying the work presented in [31].

In Section 5.1 of the present work, I have proposed a double argumentation leading precisely to the conclusion that the mean (macroscopic) connection should *not* be considered to be the average of the original (microscopic) connection; in other words, the arguments presented in Section 5.1 of this article go against the supposition which serves as a basis to [31]. Since [31] does not justify this basic supposition, and thus provides no material which could be considered as a counterargumentation to Section 5.1, I think it warranted to say that the new approach proposed in the present article is at least as physically meaningful and justified as the older one developed in [31].

7.3 Possible extensions

I would like now to mention some possible and desirable extensions to this work. The general averaging procedure introduced in this article should be applied to cosmology. What kind of equation of state for the vacuum (in the sense discussed in Sect. 6.2) do realistic cosmological models predict? This question will probably be best answered numerically. In particular, is it possible, in the light of the formalism introduced in the present article, to interpret the non-vanishing of the cosmological constant as a purely statistical effect?

Another point of possible interest. Unless one restricts the discussion ab initio to ensembles corresponding to a macro-history whose macroscopic (mean) metric has $(+,-,-,-)$ signature, one might be confronted with a mean metric with a signature different from $(+,-,-,-)$. In other words, the averaging does not generally conserve the signature. This suggests that the signature of space-time might actually be scale-dependent. This could of course have many interesting consequences which will hopefully be explored in subsequent publications.

As could perhaps be expected, our definition of what a mean gravitational field is also admits an interpretation in terms of mean geodesic motion. This will be explored separately in a forthcoming article.

Naturally, the work presented here should also be extended to include Klein-Gordon, Dirac and Proca fields as possible matter in space-time; these should be considered first as classical fields and then, if possible, as quantum fields. Linked to such an extension is a complete presentation of the averaging procedure presented in this article in group theoretical terms, viewing Einstein's relativity as a gauge theory. This will be done in a future paper and will shed new interesting light on various topics which have been just broached in the present publication. In a similar vein, a good understanding of what a mean gravitational field is should contribute to a better control of quantum field theory on curved space-time backgrounds and might even open new vistas on how General Relativity emerges as a mean field theory from a purely quantum description of gravitational phenomena.

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